STABILITY AND MONOTONICITY OF LOTKA-VOLTERRA TYPE OPERATORS

FARRUKH MUKHAMEDOV AND MANSOOR SABUROV

ABSTRACT. In the present paper, we study Lotka-Volterra (LV) type operators defined in finite dimensional simplex. We prove that any LV type operator is a surjection of the simplex. After, we introduce a new class of LV-type operators, called MLV type. We prove convergence of their trajectories and study certain its properties. Moreover, we show that such kind of operators have totaly different behavior than **f**-monotone LV type operators.

1. INTRODUCTION

Lotka-Volterra (LV) systems typically model the time evolution of conflicting species in biology [1, 37]. They have been largely studied starting with Lotka [17] and Volterra [38]. There are many other natural phenomena modelled by LV systems (see [32]). On the other hand, the use of LV discrete-time systems is a well-known subject of applied mathematics [19]. They were first introduced in a biomathematical context by Moran [22], and later popularized by May and collaborators [20, 21]. Since then, LV systems have proved to be a rich source of analysis for the investigation of dynamical properties and modelling in different domains, not only population dynamics [6, 2, 13, 18, 26], but also physics [28, 34], economy [4], mathematics [7, 14, 19, 32, 36, 35]. Typically in all these applications, the LV systems are taken quadratic. It is natural to investigate non-quadratic LV systems. In [10] were introduced, generalizing the LV systems, to model the interaction among biochemical populations. Cubic polynomial LV vector fields have appeared explicitly modelling certain phenomena arising in oscillating chemical reactions as the socalled Lotka-Volterra-Brusselator (see [5]) and in well-known predator-prey models that give rise to periodic variations in the populations (see [3, 16], etc). In [9] the global phase portraits in the Poincare disc of the cubic polynomial vector fields of LV type having a rational first integral of degree 2 is classified. There, the linearizability problem for the two-dimensional planar the cubic polynomial vector fields of LV type having a rational first integral of degree 2, is investigated. The necessary and sufficient conditions for the linearizability of this system are found.

Recently, the family of discrete-time systems termed quasipolynomial (QP) has attracted some attention in the literature [11]. In this context, it is worth noting that the interest of QP discrete systems arises from several different features. In the first place, they constitute a wide generalization of LV models. However, LV systems are not just a particular case of QP ones but play a central, in fact canonical

role in the QP framework, as will be appreciated in what is to follow. In [12] the quasipolynomial (QP) generalization of LV discrete-time systems is considered. Use of the QP formalism is made for the investigation of various global dynamical properties of QP discrete-time systems including permanence, attractivity, dissipativity and chaos. The results obtained generalize previously known criteria for discrete LV models.

In [25] it is established new sufficient conditions for global asymptotic stability of the positive equilibrium in some LV-type discrete models. Applying the former results [24] on sufficient conditions for the persistence of nonautonomous discrete LV systems, conditions for the persistence of the above autonomous system is obtained, and extending a similar technique to use a nonnegative Lyapunov like function offered by [30], new conditions for global asymptotic stability of the positive equilibrium is found.

The mentioned papers show importance the study of limiting behavior of discrete LV type operators. Therefore, in [8] **f**-monotone LV type operators on the simplex have been defined. It was proved the existence of Lyapunov functions for such operators which allowed to study limiting behaviors ones. Continuing the previous investigations, in the present paper, we first show that any LV type operator is a surjection of the simplex. After, we introduce a new class of LV type operators, called MLV type, on the simplex. We shall prove convergence of their trajectories and study certain its properties. Moreover, we show that such kind of operators have totaly different behavior than **f**-monotone LV type operators.

2. Preliminaries

Let

$$S^{m-1} = \left\{ x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : \sum_{k=1}^{m} x_k = 1, x_k \geq 0 \right\}$$

be the $(m-1)-$dimensional simplex. One can see that the points $e_k = (\delta_{1k}, \delta_{2k}, \dots, \delta_{mk})$ are the extremal points of the simplex S^{m-1}, where δ_{ik} is the Kronecker's symbol.

Let $I = \{1, 2, \dots, m\}$ and α be an arbitrary subset of I. The set

$$\Gamma_\alpha = \{x \in S^{m-1} : x_k = 0, \ k \notin \alpha\}$$

is called *a face* of the simplex. *A relatively interior* $ri\Gamma_\alpha$ of the face Γ_α is defined by

$$ri\Gamma_\alpha = \{x \in \Gamma_\alpha : x_k > 0, k \in \alpha\}.$$

The center of the face Γ_α is defined by

$$\left(0, \dots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_1}, 0, \dots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_2}, 0, \dots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_r}, 0, \dots, 0 \right)$$

where $\alpha = \{i_1, i_2, \dots, i_r\}$, $i_1 < i_2 < \cdots < i_r$ and $|\alpha|$ is the cardinality of a set α.

Given a mapping $\mathbf{f} : x \in S^{m-1} \to (f_1(x), f_2(x), \dots, f_m(x)) \in \mathbb{R}^m$ in what follows, we are interested in the following operator defined by

$$(Vx)_k = x_k(1 + f_k(x)), \ k = \overline{1, m} \ \ x \in S^{m-1}. \tag{2.1}$$

Proposition 2.1. *Let V be an operator given by* (2.1). *The following conditions are equivalent:*

(i) *The operator V is continuous in S^{m-1} and $V(S^{m-1}) \subset S^{m-1}$. Moreover, $V(ri\Gamma_\alpha) \subset ri\Gamma_\alpha$ for all $\alpha \subset I$.*

(ii) *The mapping $\mathbf{f} \equiv (f_1, f_2, \ldots, f_m) : S^{m-1} \to \mathbb{R}^m$ satisfies the following conditions:*

1^0 $\mathbf{f}$ *is continuous in S^{m-1};*

2^0 *for every $x \in S^{m-1}$ one has $f_k(x) \geq -1$, for all $k = \overline{1, m}$;*

3^0 *for every $x \in S^{m-1}$ one has $\sum_{k=1}^{m} x_k f_k(x) = 0$;*

4^0 *for every $\alpha \subset I$ one holds $f_k(x) > -1$ for all $x \in ri\Gamma_\alpha$ and $k \in \alpha$.*

Proof. $(i) \Rightarrow (ii)$. The continuity of V implies 1^0. Take $x \in S^{m-1}$ and it yields that (a) $(Vx)_k \geq 0$; (b) $\sum_{k=1}^{m} (Vx)_k = 1$. Hence, from (a) it follows that $x_k(1 + f_k(x)) \geq 0$ which implies 2^0. From (b) one has

$$\sum_{k=1}^{m} x_k + \sum_{k=1}^{m} x_k f_k(x) = 1$$

which immediately yields 3^0.

Let $x \in ri\Gamma_\alpha$, then $Vx \in ri\Gamma_\alpha$ which with (2.1) and $x_k > 0$ for all $k \in \alpha$ implies that $f_k(x) > -1$ for all $k \in \alpha$ this means 4^0.

The implication $(ii) \Rightarrow (i)$ is evident. $\qquad\square$

We say that an operator V defined by (2.1) is *Lotka-Volterra (LV) type* if one of the conditions of Proposition 2.1 is satisfied. The corresponding mapping $\mathbf{f}$ is called *generating mapping* for V. From Proposition 2.1 we immediately infer that any LV type operator maps the simplex S^{m-1} into itself. By $\mathcal{V}$ we denote the set of all LV type operators.

Note that first such kind of operators were considered in [8] and there were proved the following theorem.

Theorem 2.2 ([8]). *Let $\boldsymbol{f} \equiv (f_1, f_2, \ldots, f_m) : \mathbb{R}^m \to \mathbb{R}^m$ be a linear mapping. Then $\boldsymbol{f}$ satisfies the conditions $1^0 - 4^0$ if and only if one has*

$$f_k(x) = \sum_{i=1}^{m} a_{ki} x_i, \quad k = \overline{1, m} \tag{2.2}$$

with

$$a_{ki} = -a_{ik}, \quad |a_{ki}| \leq 1 \quad \forall \, k, i = \overline{1, m}.$$

Remark 2.3. *Note that LV type operator $V : S^{m-1} \to S^{m-1}$ with generating mapping given by* (2.2), *i.e.*

$$(Vx)_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k = \overline{1, m} \tag{2.3}$$

where $a_{ki} = -a_{ik}$, $|a_{ki}| \leq 1$, for all $k, i = \overline{1, m}$, is called quadratic volterrian operators. By $\mathcal{QV}$ we denote the set of all quadratic volterrian operators. In [7, 23, 36, 39] limiting behavior of quadratic volterrian operators were studied.

Given $x^0 \in S^{m-1}$, then the sequence $\{x^0, Vx^0, V^2x^0, \cdots, V^nx^0, \cdots\}$ is called *a trajectory of V starting from the point* x^0, where $V^{n+1}x^0 = V(V^nx^0), n = 1, 2, \ldots$ By $\omega(x^0)$ we denote the set of all limiting points of such a trajectory.

A point $x \in S^{m-1}$ is called *fixed* if $Vx = x$ and by $Fix(V)$ we denote the set of all fixed points of V. A point $x \in S^{m-1}$ is called *r-periodic* if $V^rx = x$ and $V^ix \neq x$ for all $i \in \overline{1, r-1}$.

Let us introduce some necessary notations taken from [31] (see also [27, 33]). Let U be a bounded open subset of $\mathbb{R}^m$. For $\mathbf{f} \in C^1(U)$ the Jacobi matrix of $\mathbf{f}$ at $x \in U$ is $\mathbf{f}'(x) = (\partial_{x_i} f_j(x))_{i,j=1}^m$ and the Jacobi determinant of $\mathbf{f}$ at $x \in U$ is

$$J_{\mathbf{f}}(x) = \det \mathbf{f}'(x).$$

Given $y \in \mathbb{R}^m$ let us set

$$D_y^r(\overline{U}) = \{\mathbf{f} \in C^r(\overline{U}) : y \notin \mathbf{f}(\partial U)\}, \quad D_y(\overline{U}) = \{\mathbf{f} \in C(\overline{U}) : y \notin \mathbf{f}(\partial U)\}.$$

A function $\mathbf{deg} : \mathbf{f} \in D_y(\overline{U}) \to \mathbf{deg}(\mathbf{f}, U, y) \in \mathbb{R}$ is called *degree of* $\mathbf{f}$ *at* y if it satisfies the following conditions:

(D1) $\mathbf{deg}(\mathbf{f}, U, y) = \mathbf{deg}(\mathbf{f} - y, U, 0)$;
(D2) $\mathbf{deg}(Id, U, y) = 1$ if $y \in U$, where Id is an identity mapping.
(D3) If U_1, U_2 are open, disjoint subsets of U such that $y \notin \mathbf{f}(\overline{U} \setminus (U_1 \cup U_2))$, then
    $\mathbf{deg}(\mathbf{f}, U, y) = \mathbf{deg}(\mathbf{f}, U_1, y) + \mathbf{deg}(\mathbf{f}, U_2, y)$;
(D4) If $H(t) = (1 - t)\mathbf{f} + t\mathbf{g} \in D_y(\overline{U})$, $t \in [0, 1]$, then $\mathbf{deg}(\mathbf{f}, U, y) = \mathbf{deg}(\mathbf{g}, U, y)$.

Theorem 2.4 ([33]). *There is a unique degree* $\mathbf{deg}$ *satisfying (D1)-(D4). Moreover,* $\mathbf{deg}(\cdot, U, y) : D_y(\overline{U}) \to \mathbb{Z}$ *is constant on each component and given* $\mathbf{f} \in D_y(\overline{U})$ *we have*

$$\mathbf{deg}(\mathbf{f}, U, y) = \sum_{x \in \widetilde{\mathbf{f}}(y)} sign J_{\widetilde{\mathbf{f}}}(x)$$

where $\widetilde{\mathbf{f}} \in D_y^2(\overline{U})$ *is in the same component of* $D_y(\overline{U})$, *say* $\|\mathbf{f} - \widetilde{\mathbf{f}}\| < \mathbf{dist}(y, \mathbf{f}(\partial U))$, *such that* $y \in RV(\mathbf{f}) = \{y \in \mathbb{R}^m | \forall x \in \mathbf{f}^{-1}(y) : J_{\mathbf{f}}(x) \neq 0\}$.

Theorem 2.5 ([15]). *If the degree of the mapping* f $: S^{m-1} \to S^{m-1}$ *is not zero then the mapping* f *is* **onto** *(i.e. surjective).*

3. Surjectivity of Lotka-Volterra type operators

In this section we shall show that all LV type operators are surjective.

Theorem 3.1. *Any LV type operator given by (2.1) maps simplex* S^{m-1} *onto itself. Namely, V is a surjection of* S^{m-1}.

Proof. Consider a family od operators $V_\varepsilon : S^{m-1} \to S^{m-1}$ given by

$$(V_\varepsilon x)_k = x_k(1 + \varepsilon f_k(x)), \quad k = \overline{1, m},$$

where $0 \leq \varepsilon \leq 1$, which homotopical connects an identity mapping $Id : S^{m-1} \to S^{m-1}$ and the LV type operator (2.1), i.e. $V_\varepsilon = (1-\varepsilon)Id + \varepsilon V$. According to Theorem 2.4, (D2) and (D4) we have $\deg(V) = \deg(Id) = 1$. Therefore, Theorem 2.5 implies that LV type operator (2.1) is a surjection of S^{m-1}. $\square$

It is well known (see [15]) that any continuous bijective mapping of a compact set to itself is homeomorphism of compact, hence we have the following

Corollary 3.2. *Any LV type operator given by (2.1) is homeomorphism of the simplex if and only if it is injective.*

Remark 3.3. *Note that any quadratic volterrian operator given by (2.3) is a homeomorphism of the simplex (see [7]). It is worth to note that not all LV type operators are homeomorphisms of the simplex (see Example 5.10).*

4. $M-$LOTKA-VOLTERRA TYPE OPERATORS.

In this section we introduce a class of LV type operators, called $M-$LV, and study their asymptotic behavior.

Given $x \in S^{m-1}$ put

$$M(x) = \{i \in I : x_i = \max_{k=\overline{1,m}} x_k\},$$

here as before $I = \{1, \ldots, m\}$.

Definition 4.1. *An LV type operator given by (2.1) is called M_1-Lotka-Volterra (for shortness $M_1 LV$) (resp. M_0-Lotka-Volterra ($M_0 LV$)) if for each $x \in S^{m-1}$ and for all $k \in M(x)$, $j = \overline{1,m}$ the functional*

$$\varphi(x) = x_k - x_j$$

is increasing (res. decreasing) along the trajectory of V starting from the point x, i.e. $\varphi(V^k x) \leq \varphi(V^{k+1} x)$, $k \geq 0$ (resp. $\varphi(V^k x) \geq \varphi(V^{k+1} x)$, $k \geq 0$)

By $\mathcal{VM}_1$ and $\mathcal{VM}_0$ we denote the sets of all $M_1 LV$ and $M_0 LV$ type operators, respectively.

Note that in [8] LV type operators with functionals of the form $\varphi(x) = \prod_{k=1}^{m} x_k^{p_k}$ has been investigated.

Remark 4.2. *It immediately follows from the definition that*

$$\mathcal{VM}_1 \cap \mathcal{VM}_0 = \{Id\},$$

where $Id : S^{m-1} \to S^{m-1}$ is an identity mapping.

Proposition 4.3. *Let V_0 and V_1 be $M_1 LV$ (resp. $M_0 LV$) type operators. Then the following conditions are satisfied:*

(i) *The operator $V_1 \circ V_0$ is $M_1 LV$ (res. $M_0 LV$) type.*

(ii) *For each $\lambda \in [0,1]$ the operator $(1-\lambda)V_0 + \lambda V_1$ is $M_1 LV$ (res. $M_0 LV$) type.*

Proof. Without loss of generality we may suppose that the operator V_0 and V_1 are $M_1 LV$ type. Then for each $x \in S^{m-1}$ and for all $k \in M(x)$, $j = \overline{1,m}$ we have

$$x_k - x_j \leq (V_0 x)_k - (V_0 x)_j \leq (V_1(V_0 x))_k - (V_1(V_0 x))_j$$

which implies that $V_1 \circ V_2 \in \mathcal{VM}_1$.

Now for all $\lambda \in [0,1]$ one finds

$$
\begin{aligned}
x_k - x_j &= (1-\lambda)(x_k - x_j) + \lambda(x_k - x_j) \\
&\leq (1-\lambda)((V_0 x)_k - (V_0 x)_j) + \lambda((V_1 x)_k - (V_1 x)_j) \\
&= ((1-\lambda)V_0 x + \lambda V_1 x)_k - ((1-\lambda)V_0 x + \lambda V_1 x)_j,
\end{aligned}
$$

that yields the required assertion.

By the similar argument one can prove the statements for the case of M_0LV type operators. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 4.4. *The sets $\mathcal{V}\mathcal{M}_1$ and $\mathcal{V}\mathcal{M}_0$ are convex.*

Let us provide some examples of M_1LV and M_0LV type operators, respectively.

Example 4.5. *Let us consider an operator $V_{\varepsilon,\ell}$ defined by*

$$(V_{\varepsilon,\ell}x)_k = x_k\left(1 + \varepsilon\left(x_k^\ell - \sum_{i=1}^m x_i^{\ell+1}\right)\right), \quad k = \overline{1,m} \tag{4.1}$$

where $0 < \varepsilon \le 1$ and $\ell \in \mathbb{N}$.

Let us first show that operator given by (4.1) is an LV type. One can immediately see that the generating mapping $\mathbf{f}_{\varepsilon,\ell} : S^{m-1} \to \mathbb{R}^m$ of $V_{\varepsilon,\ell}$ is given by

$$\mathbf{f}_{\varepsilon,\ell}(x) = \left(\varepsilon\left(x_1^\ell - \sum_{i=1}^m x_i^{\ell+1}\right), \varepsilon\left(x_2^\ell - \sum_{i=1}^m x_i^{\ell+1}\right), \cdots, \varepsilon\left(x_m^\ell - \sum_{i=1}^m x_i^{\ell+1}\right)\right)$$

which is obviously continuous. The inequality

$$(\mathbf{f}_{\varepsilon,\ell}(x))_k + 1 \;\; = \;\; \varepsilon\left(x_k^\ell + \sum_{i=1}^m x_i(1 - x_i^\ell)\right) + 1 - \varepsilon \ge 0 \tag{4.2}$$

shows that $(\mathbf{f}_{\varepsilon,\ell}(x))_k \ge -1$, for any $x \in S^{m-1}$ and $k = \overline{1,m}$.

One can see that

$$\sum_{k=1}^m x_k(\mathbf{f}_{\varepsilon,\ell}(x))_k = \varepsilon\left(\sum_{k=1}^m x_k^{\ell+1} - \sum_{i=1}^m x_i^{\ell+1}\sum_{k=1}^m x_k\right) = 0.$$

Take any subset $\alpha \subset I$, then the inequality (4.2) implies that $(\mathbf{f}_{\varepsilon,\ell}(x))_k > -1$, for every $x \in \mathrm{ri}\Gamma_\alpha$ and $k \in \alpha$. This means that $V_{\varepsilon,\ell}$ is a Lotka-Volterra type operator.

Now let us establish that $V_{\varepsilon,\ell}$ is M_1LV type. Indeed, take any $x \in S^{m-1}$ and one easily finds that

$$(V_{\varepsilon,\ell}x)_k - (V_{\varepsilon,\ell}x)_j = (x_k - x_j)\left(1 + \varepsilon\sum_{r=0}^\ell x_k^{\ell-r}x_j^r - \varepsilon\sum_{i=1}^m x_i^{\ell+1}\right). \tag{4.3}$$

On the other hand, the following relation

$$1 + \varepsilon\sum_{r=0}^l x_k^{\ell-r}x_j^r - \varepsilon\sum_{i=1}^m x_i^{\ell+1} = 1 - \varepsilon + \varepsilon\sum_{r=0}^\ell x_k^{\ell-r}x_j^r + \varepsilon\sum_{i=1}^m x_i(1 - x_i^\ell) \ge 0 \tag{4.4}$$

with (4.3) implies that

$$sign((V_{\varepsilon,\ell}x)_k - (V_{\varepsilon,l}x)_j) = sign(x_k - x_j),$$

which means

$$M(V_{\varepsilon,\ell}x) = M(x).$$

Take $k \in M(x)$, then we have

$$\varepsilon \sum_{r=0}^{\ell} x_k^{\ell-r} x_j^r - \varepsilon \sum_{i=1}^{m} x_i^{\ell+1} = \varepsilon \sum_{i=1}^{m} x_i(x_k^{\ell} - x_i^{\ell}) + \varepsilon \sum_{r=1}^{\ell} x_k^{\ell-r} x_j^r \geq 0, \qquad (4.5)$$

for all $j = \overline{1, m}$. Hence, from (4.3), (4.5) we obtain

$$x_k - x_j \leq (V_{\varepsilon,\ell} x)_k - (V_{\varepsilon,\ell} x)_j$$

for every $k \in M(x)$ and $j \in I$.

By the similar argument used in Example 4.5, we can show that the following operator $W_{\varepsilon,\ell}$ is M_0LV type.

Example 4.6. *Let us consider an operator $W_{\varepsilon,\ell} : S^{m-1} \to S^{m-1}$ defined by*

$$(W_{\varepsilon,\ell} x)_k = x_k \left(1 + \varepsilon \left(\sum_{i=1}^{m} x_i^{\ell+1} - x_k^{\ell} \right) \right), k = \overline{1, m} \qquad (4.6)$$

where $0 < \varepsilon \leq 1$ and $\ell \in \mathbb{N}$.

Observe that by means of the provided examples and Proposition 4.3 one can construct lots of nontrivial examples of M_1LV and M_0LV type operators, respectively.

To study stability properties of M_0LV and M_1LV type operators we need the following auxiliary result.

Lemma 4.7. *If for a sequence $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and some $k \in I$ the limits*

$$\lim_{n\to\infty} \left(x_k^{(n)} - x_j^{(n)} \right), \quad \forall \ j = \overline{1, m}, \qquad (4.7)$$

exist, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)})$, then the sequence $\{x^{(n)}\}_{n=0}^{\infty}$ converges.

Proof. The convergence of the sequences $\left\{ x_k^{(n)} - x_j^{(n)} \right\}_{n=0}^{\infty}$, for all $j = \overline{1, m}$, implies the convergence of a sequence $\left\{ \sum_{j=1}^{m} \left(x_k^{(n)} - x_j^{(n)} \right) \right\}_{n=0}^{\infty}$. Then the equality

$$m x_k^{(n)} = \sum_{j=1}^{m} \left(x_k^{(n)} - x_j^{(n)} \right) + \sum_{j=1}^{m} x_j^{(n)} = \sum_{j=1}^{m} \left(x_k^{(n)} - x_j^{(n)} \right) + 1.$$

implies the convergence of the sequence $\{x_k^{(n)}\}_{n=0}^{\infty}$. From (4.7) we obtain the convergence of $\{x_j^{(n)}\}_{n=0}^{\infty}$, for all $j = \overline{1, m}$ which yields the convergence of $\{x^{(n)}\}_{n=0}^{\infty}$. $\square$

Now we are ready to prove stability property of M_0LV and M_1LV type operators.

Theorem 4.8. *Let V be a M_1LV (resp. M_0LV) type operator. Then the trajectory $\{V^n x\}_{n=0}^{\infty}$ converges for every $x \in S^{m-1}$, i.e. $\omega(x)$ is a single point and $\omega(x) \in Fix(V)$.*

Proof. Let V be a M_1LV type operator. Then for some $k \in M(x)$ and all $j = \overline{1, m}$ we have

$$x_k - x_j \leq (Vx)_k - (Vx)_j \leq \cdots \leq (V^n x)_k - (V^n x)_j \leq \cdots \leq 1$$

Therefore the sequence $\{(V^n x)_k - (V^n x)_j\}_{n=0}^{\infty}$ converges. It follows from Lemma 4.7 that the trajectory $\{V^n x\}_{n=0}^{\infty}$ converges.

By the similar way the statement can be proved for a M_0LV case. $\qquad \square$

Lemma 4.9. *Let V be a $M_1 LV$ type operator. Then for every $x \in S^{m-1}$ and for all $n \in \mathbb{N}$ one has*

$$M(V^n x) = M(x).$$

Moreover, if there exists a limit $\lim_{n \to \infty} V^n x = x^*$ then $M(x^*) = M(x)$.

Proof. Let $k \in M(x)$. Since V is M_1LV type then for every $j = \overline{1, m}$ one has

$$0 \leq x_k - x_j \leq (Vx)_k - (Vx)_j \leq \cdots \leq (V^n x)_k - (V^n x)_j \leq \cdots \qquad (4.8)$$

which implies that $k \in M(V^n x)$ i.e.

$$M(x) \subset M(V^n x). \qquad (4.9)$$

Now we show the inclusion $M(V^n x) \subset M(x)$. Assume from the contrary, i.e. there is $k_0 \in M(V^n x)$ that $k_0 \notin M(x)$. Now take any $k_1 \in M(x)$, then from (4.9) we infer that $k_1 \in M(V^n x)$ which means $(V^n x)_{k_1} - (V^n x)_{k_0} = 0$. On the other hand, from (4.8),(4.9) we have that

$$0 < x_{k_1} - x_{k_0} \leq (Vx)_{k_1} - (Vx)_{k_0} \leq \cdots \leq (V^n x)_{k_1} - (V^n x)_{k_0} = 0$$

which is a contradiction, hence $M(V^n x) \subset M(x)$. Thus, we have $M(V^n x) = M(x)$, for any $n \in \mathbb{N}$.

Now assume that $\{V^n x\}_{n=0}^{\infty}$ converges to x^*. Then from (4.8) one has

$$x_k - x_j \leq x_k^* - x_j^* \qquad (4.10)$$

for all $k \in M(x)$ and $j = \overline{1, m}$. Then (4.10) yields that

$$M(x) \subset M(x^*). \qquad (4.11)$$

Now we show the inverse inclusion $M(x^*) \subset M(x)$. Assume from the contrary, i.e. there is $k_0 \in M(x^*)$ that $k_0 \notin M(x)$. Then we use the same argument as above, i.e. for any $k_1 \in M(x)$ it follows from (4.10), (4.11) that

$$0 < x_{k_1} - x_{k_0} \leq x_{k_1}^* - x_{k_0}^* = 0.$$

Again, the last contradiction shows that $M(x^*) \subset M(x)$ or $M(x^*) = M(x)$. $\qquad \square$

Remark 4.10. *Note that in general a similar result as Lemma 4.9 is not satisfied for $M_0 LV$ type operators (see Observation 5.8).*

Theorem 4.11. *Let V be an $M_1 LV$ type operator. Then the centers of all faces of the simplex are fixed points of V and*

$$|Fix(V)| \geq 2^m - 1, \qquad (4.12)$$

here as before $|A|$ stands for the cardinality of a set A.

Proof. Let V be an M_1LV type operator and $x^0 = (x_1^0, \ldots, x_m^0)$ be the center of the face Γ_α, i.e.

$$x_k^0 = \begin{cases} \frac{1}{|\alpha|}, & k \in \alpha \\ 0, & k \notin \alpha. \end{cases}$$

where $\alpha \subset I$.

It is clear that $M(x^0) = \alpha$. According to Theorem 4.8 the trajectory $\{V^n x^0\}_{n=0}^\infty$ converges to some point x^* which is a fixed point of V. Since the face Γ_α is invariant w.r.t. V then $x^* \in \Gamma_\alpha$. According to Lemma 4.9 we have

$$M(x^*) = M(x^0) = \alpha. \tag{4.13}$$

which means $x^* \in ri\Gamma_\alpha$. On the other hand, it follows from (4.13) that all non null coordinates of x^* are maximal, it means $x^* = x^0$. So, x^0 is a fixed point of V.

It is clear that the number of faces of the simplex is

$$\sum_{i=1}^m C_m^i = 2^m - 1,$$

so we have (4.12). $\qquad\square$

Remark 4.12. *Note that the operator $V_{\varepsilon,\ell}$ given by (4.1) was first considered in [29], in a particular case, when $\varepsilon = 1$, $l = 1$. There, it was established that for every $x^0 \in S^{m-1}$ the trajectory $\{V_{1,1}^n x^0\}_{n=0}^\infty$ starting from any $x^0 \in S^{m-1}$ always converges. Since the operator (4.1) is also $M_1 LV$ type then according to Theorem 4.8 for every $x^0 \in S^{m-1}$ the trajectory $\{V_{\varepsilon,\ell}^n x^0\}_{n=0}^\infty$ always converges for all $0 < \varepsilon \le 1$ and $\ell \in \mathbb{N}$.*

Now for the particular operator $V_{\varepsilon,\ell}$ we are going to find a limit point of the trajectory $\{V_{\varepsilon,l}^n x^0\}_{n=0}^\infty$.

Observation 4.13. *Let $x^0 \in S^{m-1}$ and $M(x^0) = \{k_1, k_2, \ldots, k_r\}$, where $k_1 < k_2 < \cdots < k_r$, then for the operator $V_{\varepsilon,\ell}$ given by (4.1) one has*

$$\lim_{n \to \infty} V_{\varepsilon,l}^n x^0 = \Big(0, \ldots, 0, \underbrace{\frac{1}{r}}_{k_1}, 0, \ldots, 0, \underbrace{\frac{1}{r}}_{k_2}, 0, \ldots, 0, \underbrace{\frac{1}{r}}_{k_r}, 0, \ldots, 0 \Big).$$

Proof. Without loss of generality we may suppose that $x^0 \in riS^{m-1}$, otherwise it is enough to consider the restriction of $V_{\varepsilon,l}$ on the face Γ_{α_0} of the simplex, where $\alpha_0 = supp(x^0)$. Here $supp(x^0) = \{i \in I : x_i^0 \ne 0\}$.

One can check that all fixed points of the operator $V_{\varepsilon,\ell}$ are only the centers of faces i.e. if $\alpha \subset I$ and $\alpha = \{i_1, i_2, \ldots, i_r\}$ then

$$Fix(V_{\varepsilon,l}) = \bigcup_{\forall \alpha \subset I} \Big(0, \ldots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_1}, 0, \ldots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_2}, 0, \ldots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_r}, 0, \ldots, 0 \Big)$$

Since the trajectory $\{V_{\varepsilon,\ell}^n x^0\}_{n=0}^\infty$ converges (see Remark 4.12) and its the limit point is a fixed point of $V_{\varepsilon,\ell}$ so we need to show that the limit point is the center of the face $\Gamma_{M(x^0)}$.

Assume from the contrary that is the limit point x^* of the trajectory $\{V_{\varepsilon,\ell}^n x^0\}_{n=0}^\infty$ is a center of another face Γ_γ, where $\gamma \neq M(x^0)$. Then it is clear $M(x^*) = \gamma$ which contradicts to Lemma 4.9. $\square$

It is interesting to know whether is there a quadratic volterrian operator (2.3) which is the M_1LV type operator.

Theorem 4.14. *A quadratic volterrian operator given by* (2.3) *is* $M_1 LV$ *type if and only if one has*

$$a_{ki} = 0, \quad \forall\, k, i = \overline{1, m},$$

i.e.

$$\mathcal{VM}_1 \cap \mathcal{QV} = \{Id\},$$

here as before $Id : S^{m-1} \to S^{m-1}$ *is an identity mapping.*

Proof. If part. Let a quadratic volterrian operator given by (2.3) be M_1LV type. According to Theorem 4.11 each center of all the face of the simplex is a fixed point of operator given by (2.3). In particular, the centers of all one dimensional faces

$$x^0 = \left(0, \ldots, 0, \underbrace{\frac{1}{2}}_{k}, 0, \ldots, 0, \underbrace{\frac{1}{2}}_{i}, 0, \ldots, 0 \right)$$

are fixed points. Then we have the following

$$\begin{cases} (Vx^0)_k = \frac{1}{2}(1 + a_{ki}\frac{1}{2}) = \frac{1}{2} \\ (Vx^0)_i = \frac{1}{2}(1 + a_{ik}\frac{1}{2}) = \frac{1}{2}, \end{cases}$$

which means $a_{ki} = 0$ for all $k, i = \overline{1, m}$.

Only if part is obvious. This completes the proof. $\square$

Problem 4.15. *At a moment similar results as Theorems 4.11, 4.14 are open for the* $M_0 LV$ *type operators.*

5. $\mathbf{f}-$ MONOTONE LOTKA-VOLTERRA TYPE OPERATORS

In this section we recall a notion of $\mathbf{f}-$monotonicity of LV type operator and study its properties.

We first recall [33], [27] that a mapping $\mathbf{f} : S^{m-1} \to \mathbb{R}^m$ is called *monotone* on the simplex S^{m-1} if for every points $x, y \in S^{m-1}$ the following condition is satisfied

$$\langle \mathbf{f}(x) - \mathbf{f}(y), x - y \rangle \geq 0, \tag{5.1}$$

where $\langle x, y \rangle$ stands for the standard scalar product in $\mathbb{R}^m$.

Observe that for a generating function $\mathbf{f}$ of LV type operator, its monotonicity can be replaced by

$$\langle \mathbf{f}(x), y \rangle + \langle x, \mathbf{f}(y) \rangle \leq 0, \qquad x, y \in S^{m-1}.$$

Definition 5.1. *An LV type operator* (2.1) *is called* $\mathbf{f}-$*monotone if its generating mapping* $\boldsymbol{f} : S^{m-1} \to \mathbb{R}^m$ *is monotone on* S^{m-1}.

By $\mathcal{FV}$ we denote the set of all $\mathbf{f}-$monotone LV type operators.

Example 5.2. *Now we are going to show that LV type operator $V_{\varepsilon,\ell}$ (see (4.1)) defined in Example 4.5 is $\mathbf{f}-$monotone.*

Indeed, we have

$$
\begin{aligned}
\langle \mathbf{f}_{\varepsilon,\ell}(x) - \mathbf{f}_{\varepsilon,\ell}(y), x - y \rangle &= -\langle x, \mathbf{f}_{\varepsilon,\ell}(y) \rangle - \langle \mathbf{f}_{\varepsilon,\ell}(x), y \rangle \\
&= -\varepsilon \sum_{k=1}^{m} x_k \left(y_k^{\ell} - \sum_{i=1}^{m} y_i^{\ell+1} \right) - \varepsilon \sum_{k=1}^{m} y_k \left(x_k^{\ell} - \sum_{i=1}^{m} x_i^{\ell+1} \right) \\
&= -\varepsilon \left(\sum_{k=1}^{m} x_k y_k^{\ell} + \sum_{k=1}^{m} x_k^{\ell} y_k - \sum_{k=1}^{m} x_k^{\ell+1} - \sum_{k=1}^{m} y_k^{\ell+1} \right) \\
&= \varepsilon \sum_{k=1}^{m} (x_k - y_k)(x_k^{\ell} - y_k^{l}) \\
&= \varepsilon \sum_{k=1}^{m} (x_k - y_k)^2 (x_k^{\ell-1} + x_k^{\ell-2} y_k + \ldots + x_k y_k^{\ell-2} + y_k^{\ell-1}) \\
&\geq 0.
\end{aligned}
$$

which shows the operator $V_{\varepsilon,\ell}$ is $\mathbf{f}-$monotone.

Note that $\mathbf{f}-$monotone LV type operators have the following properties.

Theorem 5.3 ([8]). *Let V be a $\mathbf{f}-$monotone LV type operator on S^{m-1} then the following assertions hold true:*

(i) *V is a homeomorphism of the simplex.*

(ii) *For any $x^0 \in riS^{m-1}$, $Vx^0 \neq x^0$ the set of all limit points $\omega(x^0)$ of the trajectory $\{V^n x^0\}_{n=0}^{\infty}$ belongs to the boundary ∂S^{m-1} of the simplex. Moreover either $|\omega(x^0)| = 1$ or $|\omega(x^0)| \geq \aleph_0$, here as before $|\omega(x^0)|$ means the cardinality of the set $\omega(x^0)$.*

(iii) *V has no periodic points (except for fixed points) .*

Remark 5.4. *If V is not $\mathbf{f}-$monotone then each of the given statements in Theorem 5.3 may not be satisfied. Indeed let us consider the following examples.*

First we note that in all provided examples below V is not $\mathbf{f}-$monotone.

Example 5.5. *Consider an operator V given by*

$$
(Vx)_k = x_k \left(x_k^2 + 3 \sum_{i=1}^{k-1} x_i - 3 \sum_{i,j=1, i<j}^{k-1} x_i x_j \right), \quad k = \overline{1, m}. \tag{5.2}
$$

Let us first show that the defined operator is LV type.
Since for all $k = \overline{1, m}$ we have

$$
\sum_{i=1}^{k-1} x_i - \sum_{i,j=1, i<j}^{k-1} x_i x_j = \sum_{i=1}^{k-2} x_i \left(1 - \sum_{j=i+1}^{k-1} x_j \right) + x_{k-1} \geq 0, \tag{5.3}
$$

which implies that $(Vx)_k \geq 0$ for any $x \in S^{m-1}$ and $k = \overline{1, m}$.

Denote

$$W_k(x) = \left(\sum_{i=k}^{m} x_i\right)^3 + 3\sum_{i=1}^{k-1} x_i \cdot \left(\sum_{i=k}^{m} x_i\right)^2 + 3\sum_{i,j=1,i\le j}^{k-1} x_i x_j \cdot \sum_{i=k}^{m} x_i, \quad k = \overline{1,m}.$$

Then for all $k = \overline{1,m}$ one has

$$W_k(x) = (Vx)_k + W_{k+1}(x), \quad k = \overline{1,m-1}, \tag{5.4}$$

$$W_m(x) = (Vx)_m. \tag{5.5}$$

Indeed,

$$
\begin{aligned}
W_k(x) &= \left(x_k + \sum_{i=k+1}^{m} x_i\right)^3 + 3\sum_{i=1}^{k-1} x_i \cdot \left(x_k + \sum_{i=k+1}^{m} x_i\right)^2 \\
&\quad + 3\sum_{i,j=1,i\le j}^{k-1} x_i x_j \cdot \left(x_k + \sum_{i=k+1}^{m} x_i\right) \\
&= x_k\left(x_k^2 + 3\sum_{i=1}^{k-1} x_i \cdot \sum_{i=k}^{m} x_i + 3\sum_{i,j=1,i\le j}^{k-1} x_i x_j\right) + W_{k+1}(x) \\
&= x_k\left(x_k^2 + 3\sum_{i=1}^{k-1} x_i - 3\left(\sum_{i=1}^{k-1} x_i\right)^2 + 3\sum_{i,j=1,i\le j}^{k-1} x_i x_j\right) + W_{k+1}(x) \\
&= (Vx)_k + W_{k+1}(x).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
W_m(x) &= x_m\left(x_m^2 + 3x_m\sum_{i=1}^{m-1} x_i + 3\sum_{i,j=1,i\le j}^{m-1} x_i x_j\right) \\
&= x_m\left(x_m^2 + 3\sum_{i=1}^{m-1} x_i - 3\left(\sum_{i=1}^{m-1} x_i\right)^2 + 3\sum_{i,j=1,i\le j}^{m-1} x_i x_j\right) \\
&= x_m\left(x_m^2 + 3\sum_{i=1}^{m-1} x_i - 3\sum_{i,j=1,i<j}^{m-1} x_i x_j\right) \\
&= (Vx)_m
\end{aligned}
$$

So from (5.4) and(5.5) we find

$$1 = \left(\sum_{i=1}^{m} x_i\right)^3 = W_1(x) = (Vx)_1 + W_2(x) = \cdots = \sum_{i=1}^{m} (Vx)_i.$$

Therefore, the operator given by (5.2) is LV type.

Now we show that the operator V is injective. Indeed, let $x,y \in S^{m-1}$ and $x \ne y$. Then there exists $k_0 \in I$ such that

$$x_{k_0} \ne y_{k_0}, \quad x_i = y_i, \quad \forall i = \overline{1, k_0 - 1}.$$

From (5.2) we find that

$$(Vx)_i = (Vy)_i, \quad i = \overline{1, k_0 - 1}.$$

The equality (5.3) with $x_i = y_i$ for all $i = \overline{1, k_0 - 1}$ yields that

$$C := \sum_{i=1}^{k_0-1} x_i - \sum_{i,j=1,i<j}^{k_0-1} x_i x_j = \sum_{i=1}^{k_0-1} y_i - \sum_{i,j=1,i<j}^{k_0-1} y_i y_j \geq 0.$$

Now consider a function

$$g(t) = t^3 + 3C \cdot t, \quad t \in [0,1],$$

which is strictly increasing on the segment $[0,1]$. Therefore, for $x_{k_0} \neq y_{k_0}$ we get

$$(Vx)_{k_0} = g(x_{k_0}) \neq g(y_{k_0}) = (Vy)_{k_0},$$

which means $Vx \neq Vy$.

According to Corollary 3.2 we can conclude that the operator V is a homeomorphism.

Let us show that (5.1) is not satisfied. From (5.2) we find the corresponding generating function $\mathbf{f}$ of V as follows

$$(\mathbf{f}(x))_k = x_k^2 + 3\sum_{i=1}^{k-1} x_i - 3\sum_{i,j=1,i<j}^{k-1} x_i x_j - 1, \quad k = \overline{1, m}.$$

For $x^0 = (1, 0, \ldots, 0)$ and $y^0 = (0, 1, 0, \ldots, 0)$ one has

$$\langle \mathbf{f}(x^0) - \mathbf{f}(y^0), x^0 - y^0 \rangle = -1 < 0,$$

which implies that V is not $\mathbf{f}$−monotone.

Remark 5.6. *Thus, the provided example shows that a LV type operator* (2.1) *to be a homeomorphism the $\mathbf{f}$−monotonicity is a sufficient condition.*

Let us provide another example for a LV type operator (2.1) which it is not $\mathbf{f}$−monotone and does not satisfy the assertion (ii) of Theorem 5.3.

Example 5.7. *Let us consider $M_0 LV$ type operator $W_{\varepsilon,\ell}$ defined in Example 4.6.*

Using the same argument of Example 5.2 we can show that $W_{\varepsilon,\ell}$ is not $\mathbf{f}$−monotone, i.e. one has if $x, y \in S^{m-1}$ with $x \neq y$ then

$$\langle \mathbf{f}_\varepsilon(x) - \mathbf{f}_\varepsilon(y), x - y \rangle < 0.$$

According to Theorem 4.8 for any $x^0 \in S^{m-1}$ the trajectory $\{W_{\varepsilon,\ell}^n x^0\}_{n=0}^\infty$ converges. Now we find its limit point.

Observation 5.8. *Let $x^0 \in ri\Gamma_\alpha$, where α is any subset of I, then a limit point $\omega(x^0)$ of the trajectory $\{W_{\varepsilon,\ell}^n x^0\}_{n=0}^\infty$ is the center of the face Γ_α.*

Proof. It is obvious that the fixed points of operator $W_{\varepsilon,\ell}$ (see (4.6)) are only the centers of all face of the simplex i.e. if $\alpha \subset I$ and $\alpha = \{i_1, i_2, \cdots, i_r\}$ then

$$Fix(W_{\varepsilon,l}) = \bigcup_{\forall \alpha \subset I} \left(0, \ldots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_1}, 0, \ldots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_2}, 0, \ldots, 0, \underbrace{\frac{1}{|\alpha|}}_{i_r}, 0, \ldots, 0\right)$$

Since the trajectory $\{W_{\varepsilon,\ell}^n x^0\}_{n=0}^\infty$ converges and the limit point of the trajectory is a fixed point of $W_{\varepsilon,\ell}$ so we need to show that the limit point is the center of the face Γ_α.

Without loss of generality we may suppose that $x^0 \in \mathrm{riS}^{m-1}$.

From (4.6) one gets

$$(W_{\varepsilon,l} x^0)_k = x_k^0 \left(1 + \varepsilon \sum_{i=1}^m \left((x_i^0)^l - (x_k^0)^l \right) x_i^0 \right), \quad k = \overline{1,m}. \tag{5.6}$$

Let

$$m(x^0) = \{ k \in I : x_k^0 = \min_{i=\overline{1,m}} x_i^0 \}$$

and take $k \in m(x^0)$. Then it follows from (5.6) that

$$0 < x_k^0 \leq (W_{\varepsilon,l} x^0)_k \leq (W_{\varepsilon,\ell}^2 x^0)_k \leq \cdots . \tag{5.7}$$

which means that

$$\lim_{n\to\infty} (W_{\varepsilon,\ell}^n x^0)_k > 0, \quad k \in m(x^0),$$

i.e. the minimal coordinate of the limit point is positive. Then the limit point belongs to riS^{m-1}. It is clear that the interior fixed point of $W_{\varepsilon,\ell}$ is only the center of the simplex, so we obtain

$$\lim_{n\to\infty} W_{\varepsilon,\ell}^n x^0 = \left(\frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m} \right).$$

$\square$

From Observation 5.8 we immediately find that (ii) of Theorem 5.3 is not satisfied for $W_{\varepsilon,\ell}$.

Remark 5.9. *The provided example shows that $\mathcal{VM}_0 \neq \mathcal{FV}$, since $W_{\varepsilon,l} \notin \mathcal{FV}$.*

Now we are going to show another example of LV type operator which is not **f**−monotone and does not satisfy assertion (iii) of Theorem 5.3.

Example 5.10. *Consider one dimensional simplex S^1 with the following decomposition*

$$S^1 = T_1 \bigcup T_2 \bigcup T_3 \bigcup T_4 \bigcup T_5,$$

where

$$T_1 = \left\{ x \in S^1 : 0 \leq x_1 \leq \frac{9}{30} \right\}, T_2 = \left\{ x \in S^1 : \frac{9}{30} \leq x_1 \leq \frac{11}{30} \right\},$$

$$T_3 = \left\{ x \in S^1 : \frac{11}{30} \leq x_1 \leq \frac{19}{30} \right\},$$

$$T_4 = \left\{ x \in S^1 : \frac{19}{30} \leq x_1 \leq \frac{21}{30} \right\}, T_5 = \left\{ x \in S^1 : \frac{21}{30} \leq x_1 \leq 1 \right\}.$$

Define an LV type operator $V : S^1 \to S^1$ by

$$\begin{cases} (Vx)_1 = x_1(1 + f_1(x_1, x_2)) \\ (Vx)_2 = x_2(1 + f_2(x_1, x_2)) \end{cases} \tag{5.8}$$

here

$$f_1(x_1, x_2) = \begin{cases} 2 & if \quad x \in T_1, \\ \dfrac{9}{10x_1} - 1 & if \quad x \in T_2, \\ \dfrac{4x_2 - 2}{x_1} & if \quad x \in T_3, \\ \dfrac{1}{10x_1} - 1 & if \quad x \in T_4, \\ -\dfrac{2x_2}{x_1} & if \quad x \in T_5, \end{cases} \qquad f_2(x_1, x_2) = \begin{cases} -\dfrac{2x_1}{x_2} & if \quad x \in T_1, \\ \dfrac{1}{10x_2} - 1 & if \quad x \in T_2, \\ \dfrac{4x_1 - 2}{x_2} & if \quad x \in T_3, \\ \dfrac{9}{10x_2} - 1 & if \quad x \in T_4, \\ 2 & if \quad x \in T_5. \end{cases}$$

One can see that

$$V(T_2) = \left(\frac{9}{10}, \frac{1}{10} \right), \quad V(T_4) = \left(\frac{1}{10}, \frac{9}{10} \right),$$

i.e. the operator given by (5.8) is not injective (moreover it is not a homeomorphism, see Corollary 3.2) this implies that V is not $\mathbf{f}$−monotone (see (i) of Theorem 5.3). But V has 2-periodic points in S^1. Indeed,

$$V^2 \left(\frac{1}{4}, \frac{3}{4} \right) = \left(\frac{1}{4}, \frac{3}{4} \right), \quad V^2 \left(\frac{3}{4}, \frac{1}{4} \right) = \left(\frac{3}{4}, \frac{1}{4} \right).$$

This establishes that the condition (iii) of Theorem 5.3 is not satisfied.

Remark 5.11. *Finally we have to strass that the set $\mathcal{VM}_1$ of $M_1 LV$ type operators and the set $\mathcal{FV}$ of $\mathbf{f}$−monotone LV type operators does not coincide i.e. $\mathcal{VM}_1 \neq \mathcal{FV}$. Indeed, consider the following*

Example 5.12. *Consider again one dimensional simplex S^1 with its following decomposition*

$$S^1 = T_1 \bigcup T_2 \bigcup T_3 \bigcup T_4,$$

where

$$T_1 = \left\{ x \in S^1 : 0 \leq x_1 \leq \frac{1}{3} \right\}, T_2 = \left\{ x \in S^1 : \frac{1}{3} \leq x_1 \leq \frac{5}{12} \right\},$$

$$T_3 = \left\{ x \in S^1 : \frac{5}{12} \leq x_1 \leq \frac{1}{2} \right\}, T_4 = \left\{ x \in S^1 : \frac{1}{2} \leq x_1 \leq 1 \right\}.$$

Define an LV type operator $V : S^1 \to S^1$ by

$$\begin{cases} (Vx)_1 = x_1(1 + f_1(x_1, x_2)) \\ (Vx)_2 = x_2(1 + f_2(x_1, x_2)) \end{cases} \tag{5.9}$$

here

$$f_1(x_1, x_2) = \begin{cases} 0 & if \quad x \in T_1, \\ \dfrac{x_2 - 2x_1}{3x_1} & if \quad x \in T_2, \\ \dfrac{x_1 - x_2}{2x_1} & if \quad x \in T_3, \\ 0 & if \quad x \in T_4, \end{cases} \qquad f_2(x_1, x_2) = \begin{cases} 0 & if \quad x \in T_1, \\ \dfrac{2x_1 - x_2}{3x_2} & if \quad x \in T_2, \\ \dfrac{x_2 - x_1}{2x_2} & if \quad x \in T_3, \\ 0 & if \quad x \in T_4, \end{cases}$$

One can show that the operator given by (5.9) is M_1LV type. It is easy to see that $V(T_2) = (\frac{1}{3}, \frac{2}{3})$, i.e. the operator is not injective (moreover it is not a homeomorphism, see Corollary 3.2) which means it is not **f**−monotone (see (i) of Theorem 5.3).

Acknowledgement

References

[1] Bernstein S.N. The solution of a mathematical problem concerning the theory of heredity. *Ucheniye-Zapiski N.-I. Kaf. Ukr. Otd. Mat.,* 1(1924), 83-115 (Russian).

[2] Basson M., Fogarty M.J., Harvesting in discrete-time predatorprey systems, *Math. Biosci.* **141** (1997) 41-74.

[3] Dimitrova Z. I., Vitanov N. K. Dynamical consequences of adaptation of the growth rates in a system of three competing populations, *J. Phys. A: Math. Gen.* **34**(2001) 7459–7473.

[4] Dohtani A., Occurrence of chaos in higher-dimensional discrete-time systems,*SIAM J.Appl. Math.* **52** (1992) 1707-1721.

[5] Farkas H., Noszticzius Z., Savage C. R., Schelly Z. Z. Two-dimensional explodators: II. Global analysis of the Lotka-Volterra-Brusselator (LVB) model, *Acta Phys. Hungar.* **66**(1990) 203–207.

[6] Fisher M.E., Goh B.S., Stability in a class of discrete-time models of interacting populations, *J. Math. Biol.* **4** (1977) 265-274.

[7] Ganikhodzhaev.R.N. Quadratic stochastic operators, Lyapunov functions and tournaments. *Russian Acad.Sci. Sbornik. Math.,* **76**(1993), 489-506.

[8] Ganikhodzhaev.R.N., Saburov M.Kh. A Generalized model of the nonlinear operators of Volterra type and Lyapunov functions. *Jour. Sib. Fed. Univ. Math and Phys.* **1**(2008), N 2, 188–196.

[9] Gine J., Romanovski V.G., Linearizability conditions for LotkaVolterra planar complex cubic systems, *J. Phys. A: Math. Theor.* **42** (2009) 225206 (15pp).

[10] Goel M. S., Maitra S. C., Montroll E.W. On the Volterra and other nonlinear models of interacting populations, *Rev. Mod. Phys.* **43**(1971) 231–276.

[11] Hernandez-Bermejo B., Brenig L., Quasipolynomial generalization of LotkaVolterra mappings, *J. Phys. A* **35** (2002) 5453-5469.

[12] Hernandez-Bermejo B., Brenig L., Some global results on quasipolynomial discrete systems, *Nonlinear Analysis: Real World Appl.* **7** (2006) 486 - 496.

[13] Hofbauer J., Hutson V., Jansen W., Coexistence for systems governed by difference equations of LotkaVolterra type, *J. Math. Biol.* **25** (1987) 553-570.

[14] Hofbauer J., Sigmund K., *Evolutionary Games and Population Dynamics*, Cambridge University Press, Cambridge, 1998.

[15] Kelley J.L. *General Topology*, Graduate Texts in Mathematics, 27, Springer-Verlag, New York - Heidelberg - Berlin, 1975.

[16] Lin J., Kahn P. B. Limit cycles in random environments, *SIAM J. Appl. Math.* **32**(1977) 260–291.

[17] Lotka A. J., Undamped oscillations derived from the law of mass action, *J. Amer. Chem. Soc.* **42**(1920), 1595–1599.

[18] Lu Z., Wang W., Permanence and global attractivity for Lotka-Volterra difference systems, *J. Math. Biol.* **39** (1999) 269-282.

[19] Lyubich Yu.I. *Mathematical structures in population genetics.* Springer-Verlag,Berlin, 1992.

[20] May R.M., Simple mathematical models with very complicated dynamics, *Nature* **261** (1976) 459-467;

[21] May R.M., Oster G.F., Bifurcations and dynamic complexity in simple ecological models, *Am. Nat.* **110** (1976) 573-599.

[22] Moran P.A.P., Some remarks on animal population dynamics, *Biometrics* **6** (1950) 250-258.

[23] Mukhamedov F., Saburov M. On homotopy of volterrian quadratic stochastic operator, *Appl. Math. & Inform. Sci.* **4**(2010) 47–62 (`arXiv:0712.2891`).

[24] Muroya Y., Persistence and global stability for discrete models of nonautonomous LotkaVolterra type, *J. Math. Anal. Appl.* 273 (2002) 492-511

[25] Muroya Y. Persistence and global stability in discrete models of LotkaVolterra type, *J. Math. Anal. Appl.* **330** (2007) 24-33.

[26] Narendra S.G., Samaresh C.M., Elliott W.M. On the Volterra and other nonlinear moldes of interacting populations, *Rev. Mod. Phys.* **43** (1971), 231–276.

[27] Nirenberg L. *Topics in Nonlinear Functional Analysis,* New York, 1974.

[28] Plank M., Losert V., Hamiltonian structures for the n-dimensional Lotka-Volterra equations, *J. Math. Phys.* **36** (1995) 3520–3543.

[29] Rozikov U.A., Hamraev A.Yu., On a cubic operator defined in finite dimensional simplex. *Ukr.Math.Jour.* **56**(2004), 1418–1427.

[30] Saito Y., Hara T., Ma W., Necessary and sufficient conditions for permanence and global stability of a LotkaVolterra system with two delays, *J. Math. Anal. Appl.* **236** (1999) 534-556.

[31] Steenrod N., Eiberleng S. *Foundations of Algabraic Topology*, Princeton, New Jersey 1958.

[32] Takeuchi Y., *Global dynamical properties of Lotka–Volterra systems*, World Scientific, 1996.

[33] Teschl G. *Nonlinear Functional Analysis,* Wien, 2004.

[34] Udwadia F.E., Raju N., Some global properties of a pair of coupled maps: quasi-symmetry, periodicity and syncronicity, *Physica D* **111** (1998) 16-26.

[35] Ulam S.M. *A collection of mathematical problems.* Interscience Publ. New York-London, 1960.

[36] Vallander S.S. On the limit behavior of iteration sequence of certain quadratic transformations. *Soviet Math.Doklady,* 13(1972), 123-126.

[37] Volterra V., Lois de fluctuation de la population de plusieurs espèces coexistant dans le même milieu, *Association Franc. Lyon* **1926** (1927), 96–98 (1926).

[38] Volterra V. *Lecons sur la theorie mathematique de la lutte pour la vie*, Gauthiers-Villars, Paris, 1931.

[39] Zakharevich M.I. On a limit behavior and ergodic hypothesis for quadratic mappings of a simplex. *Russian Math.Surveys,* 33(1978),207-208 (Russian).

FARRUKH MUKHAMEDOV, DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. BOX, 141, 25710, KUANTAN, PAHANG, MALAYSIA

E-mail address: `far75m@yandex.ru,farrukh_m@iiu.edu.my`

MANSOOR SABUROV, DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCE, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. BOX, 141, 25710, KUANTAN, PAHANG, MALAYSIA

E-mail address: `msaburov@gmail.com`